School Texts in Mathematics

A J Sanasam
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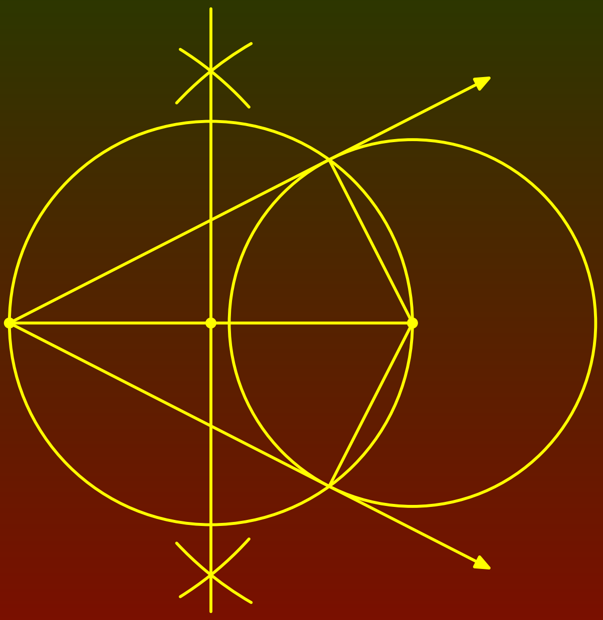
Mathematics X

Fourth Edition



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To teachers who inspire the young minds of tomorrow

School Texts in Mathematics

School Texts in Mathematics (STM), published by Lousing Chaphu, is a meticulously crafted series designed to bridge the gap between passive study and creative understanding in mathematics. Aimed at school students, this comprehensive guide covers a wide range of topics, fostering a deep and analytical grasp of mathematical concepts. Each book in the series is carefully written in LaTeX, ensuring precision and clarity in mathematical notation and presentation. The series emphasizes problem-solving skills through numerous examples and exercises. The books are versatile, serving both as textbooks for classroom use and as resources for individual study, making them invaluable for students seeking to excel in mathematics. Some books in this series are Mathematics X, Mathematics X: Previous Years' Questions and Answers and Higher Mathematics X.

Preface

An old French mathematician said: "A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street."

— David Hilbert, *International Congress of Mathematicians, Paris, 1900*

While mathematics isn't always easy to explain, our goal in this book remains unchanged: to make the subject more approachable and engaging for readers.

This fourth edition continues to offer a wide range of solved and unsolved exercises, as well as selected problems from mathematical olympiads, giving readers a glimpse into the problem-solving mindset needed in competitions. We've retained the stories, fun facts, and historical insights that help connect abstract concepts to real-world experiences.

A new chapter on trading and demat accounts has been included, aligned with the latest BSEM syllabus. Financial literacy is vital for making informed decisions about money, investments, and financial risks, promoting greater financial stability and independence. This edition includes significant updates, such as refined explanations, updated problem sets, and corrected typographical errors.

We hope this edition will inspire readers to explore the fascinating world of mathematics and develop a deeper appreciation for its elegance and applications. As always, we welcome any feedback and suggestions for further improvements in future editions.

A J Sanasam
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Thoubal
September 2024

Contents

Chapter 1

Number System

"Begin at the beginning," the King said gravely, "and go on till you come to the end: then stop."

— Lewis Carroll, *Alice in Wonderland*

A number is an abstract (mathematical) object used to count, measure, label, etc. A number system is a writing system to represent numbers. The set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers, etc., are some commonly used number systems.

Divisibility

Definition 1.1 (Divisibility). An integer a is said to divide an integer b if there exists an integer c such that $b = ac$. In this case, we say *a divides b* or *b is divisible by a* or *a is a divisor* (or *factor*) *of b* or *b is a multiple of a* and we write $a \mid b$. We write $a \nmid b$ if a does not divide b. The notation $a \mid b$ should not be confused with the fraction a/b.

Definition 1.2 (Prime number). An integer is prime if it is greater than 1 and it has exactly two distinct positive factors viz. 1 and the number itself. Equivalently, an integer is prime if it is greater than 1 and whenever it divides a product, it divides at least one of the factors.

Definition 1.3 (Composite number). An integer is composite if it has more than two distinct positive factors.

Remark. It should be noted that 1 is neither a prime nor a composite, while 2 is the only even prime.

Definition 1.4 (HCF or GCD). The *highest common factor* or *greatest common divisor* of two integers a and b is the integer d satisfying the following properties.

(a) d is non-negative, $(d \geq 0)$,

(b) d divides both a and b, $(d \mid a \text{ and } d \mid b)$,

(c) every common divisor of a and b divides d, $(e \mid a,\ e \mid b \implies e \mid d)$.

The HCF of a and b is denoted by $\gcd(a, b)$ or simply by (a, b). By definition, $(a, b) = 0$ if and only if $a = b = 0$. Otherwise $(a, b) \geq 1$.

Remark. Many authors define the HCF for two integers not both zero.

Definition 1.5 (Relatively prime or coprime). Two integers a and b are said to be relatively prime (or coprime, or prime to each other) if $(a, b) = 1$.

In other words, a and b are relatively prime if and only if their only common divisors are ± 1.

Definition 1.6 (LCM). The *least common multiple* of two integers a and b is the integer d satisfying the following properties.

(a) d is non-negative, $(d \geq 0)$,

(b) d is divisible by both a and b, $(a \mid d \text{ and } b \mid d)$,

(c) d divides every common multiple of a and b, $(a \mid e,\ b \mid e \implies d \mid e)$.

The LCM of a and b is denoted by $\operatorname{lcm}(a, b)$ or $[a, b]$. By definition, $[a, b] = 0$ if and only if $a = 0$ or $b = 0$. Otherwise $[a, b] \geq 1$.

Remark. Most authors define the LCM for two non-zero integers only.

Theorem 1.1. *The product of two positive integers is equal to the product of their HCF and LCM. In general, $(a, b)[a, b] = |ab|$ for any $a, b \in \mathbb{Z}$.*

Note. The product of three (or more) positive integers need not be equal to the product of their HCF and LCM. However, the following formulae hold good for *three* positive integers a, b, c.

$$(a, b, c) = \frac{abc[a, b, c]}{[a, b][b, c][c, a]} \quad \text{and} \quad [a, b, c] = \frac{abc(a, b, c)}{(a, b)(b, c)(c, a)}.$$

Lemma. A lemma is a provable statement used in proving another statement.
Algorithm. An algorithm is a well defined sequence of steps forming a process for solving a given problem.

Euclid's Division Lemma

Theorem 1.2 (Euclid's division lemma). *Let a and b be any two integers with $b > 0$. Then there exist unique integers q and r such that $a = bq + r$ and $0 \leq r < b$.*

The integer q is called the quotient of a with respect to b and the integer r is called the remainder of a with respect to b. Euclid's division lemma is also known as the division algorithm.

Example 1. Show that the sum of the squares of two odd integers is of the form $4k + 2$.

Solution. Let m and n be any two odd integers. We know that every odd integer is of the form $2q + 1$, for some integer q. So, $m = 2a + 1$ and $n = 2b + 1$ for some integers a and b.

$$\begin{aligned} \therefore\ m^2 + n^2 &= (2a+1)^2 + (2b+1)^2 \\ &= 4a^2 + 4a + 1 + 4b^2 + 4b + 1 \\ &= 4(a^2 + a + b^2 + b) + 2 \\ &= 4k + 2, \quad \text{where } k = a^2 + a + b^2 + b \in \mathbb{Z}. \end{aligned}$$

This shows that the sum of the squares of two odd integers is of the form $4k + 2$. □

Exercise 2. Prove that every prime number greater than 3 is of the form $6k + 1$ or $6k + 5$, where k is some integer.

Hint. $6k + r$ is divisible by 2 if $r = 0, 2, 4$ and is divisible by 3 if $r = 3$.

Exercise 3. The square of any positive integer cannot be of the form $3m + 2$, where m is a natural number. Justify.

Euclid's Algorithm

Euclid's algorithm for finding the HCF of two given positive integers.

Step 1. Find the quotient and remainder of the division of the greater number by the smaller.

Step 2. If the remainder is zero, then the divisor is the HCF.

Step 3. Else, taking the previous remainder as the new divisor and the previous divisor as the new dividend, find the quotient and the remainder.

Step 4. Continue the process till the remainder is zero. The last divisor is the required HCF.

Remark. Although Euclid's algorithm is stated for only positive integers, it can be extended for all non-zero integers. Euclid's algorithm works because of the following result.

Theorem 1.3. *If* $a = bq + r$*, then* $(a, b) = (b, r)$*.*

Proof[1]**.** If $(a, b) = d$, then $d \mid r$. If $c \mid b$ and $c \mid r$, then $c \mid a$. Now, $c \mid a,\ c \mid b \implies c \mid d$. Thus, d divides both b and r, and any common divisor c of b and r also divides d. Hence, $(b, r) = d$.

Example 4. Let a and b be two positive integers where $a > b$. Prove that the last divisor (or the last non-zero remainder) in the Euclid's algorithm for a and b is the HCF of a and b.

Solution. Applying Euclid's division lemma successively, we get the following series of relations.

$$
\begin{aligned}
a &= bq + r_1, & 0 &< r_1 < b, & (1)\\
b &= r_1 q_1 + r_2, & 0 &< r_2 < r_1, & (2)\\
r_1 &= r_2 q_2 + r_3, & 0 &< r_3 < r_2, & (3)\\
&\vdots & &\vdots & \\
r_{n-2} &= r_{n-1} q_{n-1} + r_n, & 0 &< r_n < r_{n-1}, & (n)\\
r_{n-1} &= r_n q_n + 0. & & & (n+1)
\end{aligned}
$$

Here, $q, q_1, \ldots, q_n,\ r_1, r_2, \ldots, r_n$ are positive integers and $a > b > r_1 > r_2 > \cdots > r_n$. The set of equations (1) to $(n+1)$ is the Euclid's algorithm for a and b.

We know that if $a = bq + r$, then $(a, b) = (b, r)$. Applying this result successively to the equations (1) to $(n + 1)$, we get

$$(a, b) = (b, r_1) = (r_1, r_2) = \cdots = (r_{n-1}, r_n) = r_n.$$

The last equality is true because r_n divides r_{n-1} as seen from the $(n+1)^{\text{th}}$ equation. Thus, we see that r_n, which is the last divisor (or the last non-zero remainder) in the Euclid's algorithm, is the HCF of a and b. □

Example 5. Using Euclid's algorithm, determine whether the numbers 20 and 12 are coprime or not.

[1]Proof is not required for HSLC Examination.

Solution. Euclid's algorithm for the two integers comprises of the following equalities.

$$\begin{aligned} 20 &= 12 \times 1 + 8, \\ 12 &= 8 \times 1 + 4, \\ 8 &= 4 \times 2 + 0. \end{aligned}$$

Here, the last divisor is 4 and hence the HCF of 20 and 12 is 4. Since the HCF is not 1, the numbers 20 and 12 are not coprime.

Example 6. Prove that the fraction $\dfrac{21n+4}{14n+3}$ is irreducible (i.e., is in lowest terms) for every natural number n. **(IMO 1959)**

Solution. A fraction $\dfrac{a}{b}$ is irreducible if $\gcd(a,b) = 1$. We shall use Euclid's algorithm to show that $(21n+4, 14n+3) = 1\ \forall\ n \in \mathbb{N}$. We have

$$\begin{aligned} 21n+4 &= (14n+3) \times 1 + (7n+1), \\ 14n+3 &= (7n+1) \times 2 + 1, \\ 7n+1 &= 1 \times (7n+1) + 0. \end{aligned}$$

Here, the last divisor is 1 and hence $(21n+4, 14n+3) = 1\ \forall\ n \in \mathbb{N}$. Consequently $\dfrac{21n+4}{14n+3}$ is irreducible for every natural number n. □

Remark. If d is a common divisor of a and b, then d divides $ax + by$ for any integers x and y. Example 6 can also be solved by using this result. If $(21n+4, 14n+3) = d$, then d divides $-2(21n+4) + 3(14n+3)$, which implies that d divides 1. Hence, $d = 1$.

Exercise 7. Prove that the fraction $\dfrac{12n+1}{30n+2}$ is irreducible for every natural number n.

Example 8. Prove that $(2^n - 1, 2^n + 1) = 1$ for any positive integer n.

Solution. If $n = 1$, then it is clear that $(2^n - 1, 2^n + 1) = (1, 3) = 1$. For $n \geq 2$, we have

$$\begin{aligned} 2^n + 1 &= (2^n - 1) \times 1 + 2, \\ 2^n - 1 &= 2 \times 2^{n-1} - 1 = 2 \times 2^{n-1} - 2 + 1 = 2 \times \left(2^{n-1} - 1\right) + 1, \\ 2 &= 1 \times 2 + 0. \end{aligned}$$

Hence, $(2^n - 1, 2^n + 1) = 1$ for $n \geq 2$. This completes the proof. □

Remark. If $d = (2^n - 1, 2^n + 1)$, then d divides $(2^n + 1) - (2^n - 1)$, i.e., d divides 2, which implies that $d = 1$ or $d = 2$. But 2 does not divide $2^n + 1$ for any positive integer n. So, d cannot be 2, and consequently $d = 1$.

Exercise 1.1

1. Using Euclid's algorithm find the HCF of

(i) 1240 and 1984,

(ii) 348 and 504,

(iii) 986 and 899,

(iv) 4216 and 1240,

(v) 10605 and 5256,

(vi) 10005 and 9269.

Solution.

(i) Euclid's algorithm for the two integers 1240 and 1984 comprises of the following equalities.

$$\begin{aligned}1984 &= 1240 \times 1 + 744,\\ 1240 &= 744 \times 1 + 496,\\ 744 &= 496 \times 1 + 248,\\ 496 &= 248 \times 2 + 0.\end{aligned}$$

The last divisor is 248 and hence the required HCF is 248.

(ii) The two given integers are 348 and 504. By Euclid's algorithm, we have

$$\begin{aligned}504 &= 348 \times 1 + 156,\\ 348 &= 156 \times 2 + 36,\\ 156 &= 36 \times 4 + 12,\\ 36 &= 12 \times 3 + 0.\end{aligned}$$

The last divisor is 12 and hence the required HCF is 12.

(iii) The two given integers are 986 and 899. By Euclid's algorithm, we have

$$\begin{aligned}986 &= 899 \times 1 + 87,\\ 899 &= 87 \times 10 + 29,\\ 87 &= 29 \times 3 + 0.\end{aligned}$$

The last divisor is 29 and hence the required HCF is 29.

(iv) The two given integers are 4216 and 1240. By Euclid's algorithm, we have

$$\begin{aligned}4216 &= 1240 \times 3 + 496,\\ 1240 &= 496 \times 2 + 248,\\ 496 &= 248 \times 2 + 0.\end{aligned}$$

The last divisor is 248 and hence the required HCF is 248.

(v) The two given integers are 10605 and 5256. By Euclid's algorithm, we have

$$\begin{aligned} 10605 &= 5256 \times 2 + 93, \\ 5256 &= 93 \times 56 + 48, \\ 93 &= 48 \times 1 + 45, \\ 48 &= 45 \times 1 + 3, \\ 45 &= 3 \times 15 + 0. \end{aligned}$$

The last divisor is 3 and hence the HCF of 10605 and 5256 is 3.

(vi) The two given integers are 10005 and 9269. By Euclid's algorithm, we have

$$\begin{aligned} 10005 &= 9269 \times 1 + 736, \\ 9269 &= 736 \times 12 + 437, \\ 736 &= 437 \times 1 + 299, \\ 437 &= 299 \times 1 + 138, \\ 299 &= 138 \times 2 + 23, \\ 138 &= 23 \times 6 + 0. \end{aligned}$$

The last divisor is 23 and hence $(10005, 9269) = 23$.

2. Show that the product of two consecutive integers is divisible by 2.

Solution. Let a, $a+1$ be the two consecutive integers. Then a is of the form $2q$ or $2q+1$ for some integer q.

If $a = 2q$, then $a(a+1) = 2q(2q+1)$, which is divisible by 2.

If $a = 2q+1$, then
$$\begin{aligned} a(a+1) &= (2q+1)(2q+1+1) \\ &= 2(2q+1)(q+1), \end{aligned}$$

which is divisible by 2. Thus, the product of two consecutive integers is divisible by 2. □

3. Show that the product of two consecutive even integers is divisible by 8.

Solution. Let $2a$ and $2a+2$ be the two consecutive even integers where a is some integer. The integer a is of the form $2q$ or $2q+1$ for some integer q.

If $a = 2q$, then
$$\begin{aligned} 2a(2a+2) &= 2(2q)(4q+2) \\ &= 8q(2q+1), \text{ which is divisible by 8.} \end{aligned}$$

If $a = 2q + 1$, then $2a(2a + 2) = 2(2q + 1)(4q + 2 + 2)$
$$= 8(2q + 1)(q + 1),$$

which is divisible by 8.

Thus, $2a(2a + 2)$ is divisible by 8 for any integer a. □

4. Show that every integer is of the form $4q$, $4q + 1$, $4q + 2$ or $4q - 1$.

Solution. By Euclid's division lemma, for any integer a, we have

$$a = 4k + r, \text{ where } k, r \in \mathbb{Z},\ 0 \leq r < 4.$$

If $r = 0, 1$ or 2, then a is of the form $4q$, $4q + 1$ or $4q + 2$.
If $r = 3$, then $a = 4k + 3 = 4(k + 1) - 1 = 4q - 1$, where $q = k + 1 \in \mathbb{Z}$.
Thus, a is of the form $4q$, $4q + 1$, $4q + 2$ or $4q - 1$. □

5. Show that the product of three consecutive integers is divisible by 6.

Solution. Let a, $a + 1$ and $a + 2$ be the three consecutive integers. The integer a is of the form $2q$ or $2q + 1$. If $a = 2q$, then

$$a(a + 1)(a + 2) = 2q(2q + 1)(2q + 2), \text{ which is divisible by 2.}$$

If $a = 2q + 1$, then

$$\begin{aligned} a(a + 1)(a + 2) &= (2q + 1)(2q + 2)(2q + 3) \\ &= 2(2q + 1)(q + 1)(2q + 3), \text{ which is divisible by 2.} \end{aligned}$$

Thus, $a(a + 1)(a + 2)$ is divisible by 2 for any integer a.

Also, the integer a is of the form $3q$, $3q + 1$ or $3q + 2$. If $a = 3q$, then

$$a(a + 1)(a + 2) = 3q(3q + 1)(3q + 2), \text{ which is divisible by 3.}$$

If $a = 3q + 1$, then

$$a(a + 1)(a + 2) = 3(3q + 1)(3q + 2)(q + 1), \text{ which is divisible by 3.}$$

If $a = 3q + 2$, then

$$a(a + 1)(a + 2) = 3(3q + 2)(q + 1)(3q + 4), \text{ which is divisible by 3.}$$

Thus, $a(a + 1)(a + 2)$ is divisible by 3 for any integer a.

It is observed that $a(a + 1)(a + 2)$ is divisible by both 2 and 3 for any integer a, and $(2, 3) = 1$. Hence, $a(a + 1)(a + 2)$ is divisible by 6. □

Remark. Q5 can be solved using the fact that any integer a is of the form $6q + r$, where $0 \leq r < 6$.

Geometrical meaning of the zeroes of a polynomial: The zeroes of a polynomial $p(x)$ are precisely the x-coordinates of the points where the graph representing $y = p(x)$ intersects the x-axis. The graph of a linear polynomial $ax + b, a \neq 0$, is a straight line. It intersects the x-axis at exactly one point. So, a linear polynomial has exactly one zero. The graph of a quadratic polynomial $ax^2 + bx + c, a \neq 0$, is a parabola. It intersects the x-axis at most at 2 points. So, a quadratic polynomial has at most 2 zeroes. In general, given a polynomial $p(x)$ of degree n, the graph of $y = p(x)$ intersects the x-axis at most at n points. So, a polynomial $p(x)$ of degree n has at most n zeroes. The following figure shows the graph of $y = x^3 - 2x^2 - x + 2$ intersecting the x-axis at three distinct points.

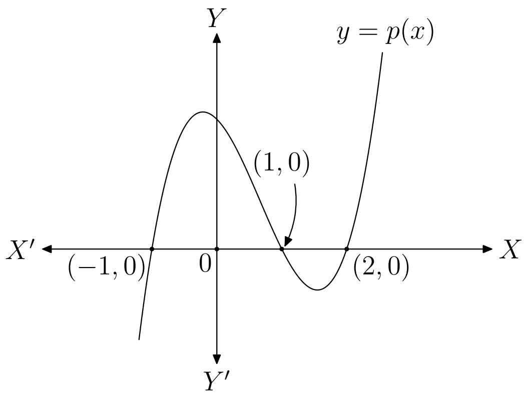

Figure 2.1: Graph of $y = p(x) = x^3 - 2x^2 - x + 2$.

Remark. The zeroes we have discussed above are the zeroes which are real. In the complex number system (see page 150), a polynomial of degree $n \geq 1$ has exactly n zeroes (counting multiplicities).

Long division process of polynomials:

Let us discuss the long division process of polynomials with the help of an example. Consider the division of $4 + 2x^2 + 3x$ by $2 + x$. We carry out the division by means of the following steps.

Step 1. We arrange the terms of the dividend and the division in the descending order of their degrees. (If any term is missing in the dividend, a zero may be used to fill in the missing term.) The dividend is $2x^2 + 3x + 4$ and the divisor is $x + 2$.

Step 2. We divide the first term of the dividend by the first term of the divisor, i.e., we divide $2x^2$ by x and we get $2x$. $2x$ is the first term of the quotient.

$$
\begin{array}{r|l}
 & \quad\;\; 2x - 1 \\ \hline
x + 2 & 2x^2 + 3x + 4 \\
 & 2x^2 + 4x \\ \hline
 & \quad\;\; -x + 4 \\
 & \quad\;\; -x - 2 \\ \hline
 & \qquad\quad 6
\end{array}
$$

Step 3. We multiply the divisor $x+2$ by $2x$ (the first term of the quotient) and obtain the product $2x^2+4x$. We subtract this product $2x^2+4x$ from the dividend $2x^2+3x+4$ and we get the remainder $-x+4$.

Step 4. We treat the remainder $-x+4$ as the new dividend, keeping the divisor the same. We divide the first term $-x$ of the new dividend by the first term of the divisor and obtain -1. -1 is the second term of the quotient.

Step 5. We multiply the divisor $x+2$ by -1 (the second term of the quotient) and subtract the product $-x-2$ from the dividend $-x+4$. This gives 6 as the remainder which will be the new dividend for the next step.

Note that steps 4 and 5 are repetitions of the steps 2 and 3 with a new dividend. The process continues until the remainder is zero or the degree of the new dividend is less than that of the divisor. In our present case, the degree of the remainder 6 (which has degree 0) is less than the degree of the divisor $x+2$ and so we stop the process here. The last remainder (i.e., the dividend at the last stage) 6 is the remainder of the division of $2x^2+3x+4$ by $x+2$. The quotient is $2x-1$ (the sum of the quotient terms obtained in steps 2 and 4). We see that $2x^2+3x+4=(x+2)(2x-1)+6$, i.e., dividend = divisor × quotient + remainder.

Theorem 2.1 (**Division algorithm for polynomials**). *For any polynomials $p(x)$ and $d(x) \neq 0$, there exist unique polynomials $q(x)$ and $r(x)$ such that $p(x) = d(x) \times q(x) + r(x)$, where either $r(x) = 0$ or degree of $r(x) <$ degree of $d(x)$.*

The polynomial $p(x)$ is the dividend and the non-zero polynomial $d(x)$ is the divisor. The polynomial $q(x)$ is called the quotient and the polynomial $r(x)$ is called the remainder.

Remark. In Chapter 1, we have defined the HCF and the LCM of two numbers. In a similar way, we can define the HCF and the LCM of two polynomials. We have also discussed the Euclid's algorithm for finding the HCF of two numbers. A similar algorithm can be used for finding the HCF of two polynomials.

Exercise 1. Find the quotient $q(x)$ and the remainder $r(x)$ when the polynomial $p(x)$ is divided by the polynomial $d(x)$ and verify the division algorithm in each of the following:

(i) $p(x) = 3x^3 - 5x^2 + 10x + 5$, $d(x) = 3x + 1$,

(ii) $p(x) = x^3 - 2x^2 - 12$, $d(x) = 3 - x$,

(iii) $p(x) = 2x^3 - 9x^2 + 25$, $d(x) = 2x - 5$,

Exercise 4. Prove that

$$a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2) = (a - b)(b - c)(c - a).$$

Exercise 5. Prove that $a^2(b + c) + b^2(c + a) + c^2(a + b) + 3abc = 0$ if $a = x - y$, $b = y - z$, $c = z - x$.

Exercise 6. Prove that $a^2(b + c) + b^2(c + a) + c^2(a + b) + 2abc = 0$ if $a = x^2 - 3xy - y^2$ and $b = y^2 + 3xy - x^2$.

Exercise 7. Find the value of $x^2(y + z) + y^2(z + x) + z^2(x + y) + 3xyz$ if $x + y + z = 23$ and $x^2 + y^2 + z^2 = 181$. (Answer. 4002.)

Exercise 8. If $x^2(y-z)+y^2(z-x)+z^2(x-y) = 6$ and $xy+yz+yx = 174$, find the value of $x^3(y^2 - z^2) + y^3(z^2 - x^2) + z^3(x^2 - y^2)$. (Answer. 1044.)

Exercise 9. If $x + y = 14$, $y + z = 15$, $z + x = 17$, find the value of $x^3 + y^3 + z^3$. (Answer. 1457.)

Exercise 10. Find the value of $x^3 + y^3 + y^3 - 3xyz$ if $x + y + z = 23$ and $xy + yz + zx = 174$. (Answer. 161.)

Example 11. Factorise $a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) - 2abc$.

Solution. The given expression is factorised as follows.

$$\begin{aligned}
&a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c)] - 2abc \\
&= a^2(b + c - a) + b^2c + ab^2 - b^3 + c^2a + bc^2 - c^3 - 2abc \\
&= a^2(b + c - a) - b^2(b + c - a) - c^2(b + c - a) + 2bc(b + c - a) \\
&= (b + c - a)(a^2 - b^2 - c^2 + 2bc) \\
&= (b + c - a)\{a^2 - (b - c)^2\} \\
&= (b + c - a)\{a - (b - c)\}\{a + (b - c)\} \\
&= (b + c - a)(c + a - b)(a + b - c).
\end{aligned}$$

Exercise 12. Determine the least positive value taken by the expression $a^3 + b^3 + c^3 - 3abc$ as a, b, c vary over all positive integers. Find also all triples (a, b, c) for which this least value is attained. (**INMO 2002**)

Hint. Use $a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a+b+c)\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$. Clearly, a, b, c are not all equal (for example, if $a = b$, then $b \neq c \neq a$). So, $a+b+c \geq 1+1+2 = 4$ and $(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0^2 + 1^2 + 1^2 = 2$. Minimum value is 4 and is attained at $(1, 1, 2)$, $(1, 2, 1)$ and $(2, 1, 1)$.

Exercise 3.1

1. Factorise the following:

(i) $x^3 + y^3 - z^3 + 3xyz$,

(ii) $a^3 - b^3 + 9ab + 27$,

(iii) $8a^3 + 27b^3 + 64c^3 - 72abc$,

(iv) $x^3 - y^3 - 125z^3 - 15xyz$,

(v) $a^6 + 5a^3 + 8$,

(vi) $x^6 + 8x^3 + 27$.

Solution.

(i) The given expression is factorised as follows.

$$\begin{aligned}
&x^3 + y^3 - z^3 + 3xyz \\
&= (x+y)^3 - 3xy(x+y) - z^3 + 3xyz \\
&\hfill [\because a^3 + b^3 = (a+b)^3 - 3ab(a+b)] \\
&= (x+y)^3 - z^3 - 3xy(x+y-z) \\
&= (x+y-z)\left\{(x+y)^2 + (x+y)z + z^2\right\} - 3xy(x+y-z) \\
&\hfill [\because a^3 - b^3 = (a-b)(a^2 + ab + b^2)] \\
&= (x+y-z)(x^2 + 2xy + y^2 + zx + yz + z^2 - 3xy) \\
&= (x+y-z)(x^2 + y^2 + z^2 - xy + yz + zx).
\end{aligned}$$

(ii) We have

$$\begin{aligned}
&a^3 - b^3 + 27 + 9ab \\
&= a^3 + (-b)^3 + 3^3 - 3 \cdot a \cdot (-b) \cdot 3 \\
&= x^3 + y^3 + z^3 - 3xyz, \text{where } x = a, y = -b, z = 3 \\
&= (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx) \\
&= (a - b + 3)\{a^2 + (-b)^2 + 3^2 - (a)(-b) - (-b)(3) - (3)(a)\} \\
&= (a - b + 3)(a^2 + b^2 + 9 + ab + 3b - 3a).
\end{aligned}$$

(iii) We have

$$\begin{aligned}
&8a^3 + 27b^3 + 64c^3 - 72abc \\
&= (2a)^3 + (3b)^3 + (4c)^3 - 3(2a)(3b)(4c) \\
&= x^3 + y^3 + z^3 - 3xyz, \text{where } x = 2a, y = 3b, z = 4c \\
&= (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx) \\
&= (2a + 3b + 4c)\{(2a)^2 + (3b)^2 + (4c)^2 - (2a)(3b) \\
&\qquad - (3b)(4c) - (4c)(2a)\} \\
&= (2a + 3b + 4c)(4a^2 + 9b^2 + 16c^2 - 6ab - 12bc - 8ca).
\end{aligned}$$

We also say that $x = \alpha$ is a solution of the quadratic equation or that α satisfies the quadratic equation. Note that a root of the quadratic equation $ax^2 + bx + c = 0$ is a zero of the polynomial $ax^2 + bx + c$ and vice-versa. A quadratic equation cannot have more than two roots.

Exercise 1. Check whether the following are quadratic equations:
(i) $x^3 + 2015 = (x - 1)^3$, **(ii)** $2x^2 + 3x - 1 = (2x + 1)(x + 5)$.
(Answer. (i) Quadratic. (ii) Not quadratic.)

Quadratic equations can be solved by (i) the method of factorisation, or by (ii) the method of completing the square, also known as the Hindu method or the Sreedharacharya's method.

Solution of a quadratic equation by factorisation:

In this method, we find the roots of the quadratic equation $ax^2+bx+c = 0$, $a \neq 0$, by factorising the polynomial $ax^2 + bx + c$ into two linear factors and then equating each factor to 0 (using corollary 1.13, page 25).

Exercise 2. Solve by the method of factorisation:
(a) $x^2 - 3x - 10 = 0$, (Answer. -2, 5.)
(b) $2x^2 - 5x + 3 = 0$, (Answer. 1, 3/2.)
(c) $100x^2 - 20x + 1 = 0$. (Answer. 1/10, 1/10.)

Exercise 3. Solve: $\dfrac{4}{x} - 3 = \dfrac{5}{2x + 3}$. (Answer. -2, 1.)

Solution of a quadratic equation by completing the square:
Consider the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$. Dividing throughout by a, we get

$$
\begin{aligned}
& x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \\
\implies & x^2 + 2 \times x \times \frac{b}{2a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \\
\implies & \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \\
\implies & x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
\implies & x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\end{aligned}
$$

Note. Completion of the square can also be done after multiplying the given equation throughout by $4a$.

Quadratic formula:

The roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Do you know? Évariste Galois was born on October 25, 1811, near Paris. The first eleven years of his life were happy. Galois, at the age of sixteen, attempted to enter the prestigious École Polytechnique, but failed in the entrance examination. Commenting on his failure, Terquem remarked, "A candidate of superior intelligence is lost with an examiner of inferior intelligence." After this, things started to go very bad for Galois. On April 2, 1829, Galois' father committed suicide. He once again tried to enter the École Polytechnique, but again failed under some rather controversial circumstances. In one last desperate effort to gain recognition, in 1831, he had sent a memoir on the general solution of equations to the Academy of Sciences. Moreover, Galois' proof was, to say least, sketchy. Poisson, after reading Galois' memoir, remarked

> We have made every effort to understand Mr. Galois' proof. His arguments are not clear enough, nor developed enough, for us to be able to judge their correctness...

Galois' paper was rejected for publication. In May 1832, Galois had a brief love affair with a young woman. He broke of the affair on May 14, and this appears to be the cause of subsequent duel that proved fatal to Galois. Galois died on May 31, 1832 at the age of 20. Fourteen years later, in 1846, Galois' work was finally published. What he proved in his paper was that for any $n \geq 5$, there is no algebraic formula, involving only the four basic arithmetic operations and the taking of roots, that gives the solutions to any polynomial equation of degree n.

Exercise 4. Solve by method of completing the square:

$$4x^2 - 2(a^2 + b^2)x + a^2b^2 = 0. \qquad \big(\text{Answer. } a^2/2,\ b^2/2.\big)$$

Exercise 5. Solve by using quadratic formula:

$$abx^2 + (b^2 - ac)x - bc = 0, \text{ where } ab \neq 0. \qquad (\text{Answer. } c/b,\ -b/a.)$$

Exercise 6. Factorise the quadratic polynomial $ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$, by using quadratic formula. State the conditions under which the polynomial can or cannot be factorised over $\mathbb{R}$.

Hint. Let $p(x) = ax^2 + bx + c$. If α and β are the roots of $p(x) = 0$, then $p(\alpha) = p(\beta) = 0$. By factor theorem, $x-\alpha$ and $x-\beta$ are the factors of $p(x)$. So, $p(x) = ax^2 + bx + c = k(x-\alpha)(x-\beta)$. Comparing the coefficients of x^2, we get $k = a$. Hence, $p(x) = a(x-\alpha)(x-\beta)$. Consequently, $ax^2 + bx + c = a\left(x + \frac{b - \sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)$. Note that $\sqrt{b^2 - 4ac}$ is real for $b^2 - 4ac \geq 0$ and not real for $b^2 - 4ac < 0$. Therefore, $p(x)$ can be factorised over $\mathbb{R}$ when $b^2 - 4ac \geq 0$ and cannot be factorised over $\mathbb{R}$ when $b^2 - 4ac < 0$.

Definition 5.3 (Discriminant). The quantity $b^2 - 4ac$ is called the discriminant of the quadratic equation $ax^2 + bx + c = 0$.

Nature of the roots of a quadratic equation:

The nature of the roots of $ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$, depends on the value of its discriminant $b^2 - 4ac$. According to the value of the discriminant, we consider the following three distinct cases.

Case I. $b^2 - 4ac > 0$.
If $b^2 - 4ac > 0$, then $\sqrt{b^2 - 4ac}$ is real and hence we get two real and distinct roots $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Further, if a, b, c are rational and $b^2 - 4ac$ is the square of a rational, then the roots are rational ($\sqrt{b^2 - 4ac}$ being rational) and if $b^2 - 4ac$ is not the square of a rational, then the roots are irrational.

Case II. $b^2 - 4ac = 0$.
If $b^2 - 4ac = 0$, then $\sqrt{b^2 - 4ac} = 0$ and so the quadratic equation has two real and equal roots, each being $-\frac{b}{2a}$.

Case III. $b^2 - 4ac < 0$.
If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is not real, since here is no real number whose square is the negative number $b^2 - 4ac$. Hence, the roots of the quadratic equation are not real.

Thus, the roots of the quadratic equation $ax^2 + bx + x = 0$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$, are

(1) real and unequal if and only if $b^2 - 4ac > 0$,

(2) real and equal if and only if $b^2 - 4ac = 0$,

(3) not real (or imaginary) and unequal if and only if $b^2 - 4ac < 0$.

Proof. In right triangle PQR, by Pythagoras theorem,

$$\begin{aligned} QR^2 &= PQ^2 + PR^2 \\ &= AB^2 + AC^2 \quad \text{(construction)} \\ &= BC^2 \quad \text{(given)} \\ \implies QR &= BC. \end{aligned}$$

Now, in $\triangle$s ABC and PQR,

$$AB = PQ, \quad BC = QR \quad \text{and} \quad AC = PR.$$

By SSS congruence, $\triangle ABC \cong \triangle PQR$. Hence, $\angle A = \angle P = 90°$. □

Do you know? Pythagorean triples are the triples (a, b, c) of positive integers such that $a^2 + b^2 = c^2$. The smallest Pythagorean triple is $(a, b, c) = (3, 4, 5)$. There are infinitely many Pythagorean triples. While studying Pythagoras theorem and Pythagorean triples, Pierre de Fermat, a French mathematician, came up with the famous **Fermat's last theorem** which states that *the equation*

$$x^n + y^n = z^n$$

has no solution in positive integers for all integers $n > 2$. In the margin of his copy of Diophantus' *Arithmetica*, he wrote

> *I have a truly marvellous demonstration of this proposition which this margin is too narrow to contain.*

Fermat's last theorem confounded the world's greatest mathematicians for more than 350 years. It was finally proved by Andrew Wiles, a British mathematician, after working in complete secrecy for seven years. His proof brings together some areas of mathematics which were previously considered completely different. Wiles' proof used many techniques of modern mathematics which were not invented at the time of Fermat. The proof which goes on for over a hundred pages was finally published in *Annals of Mathematics* in May 1995.

Exercise 56. Prove that the sum of the squares of the sides of a rhombus is equal to the sum of the squares of its diagonals.

Exercise 57. A ladder 10 m long reaches the top of a vertical wall 8 m above the ground. Find the distance of the foot of the ladder from base of the wall. (Answer. 6 m.)

Exercise 58. Two poles of heights 6 m and 11 m stand on a plane ground. If the distance between the feet of the poles is 12 m, find the distance between their tops. (Answer. 13 m.)

Exercise 59. In an acute $\triangle ABC$, $\angle B$ is an acute angle and D is a point on BC such that $AD \perp BC$. Prove that $AC^2 = AB^2 + BC^2 - 2BC \times BD$.

Hint. $AC^2 = AD^2 + (BC - BD)^2$ and $AB^2 = AD^2 + BD^2$.

Exercise 60. In an obtuse $\triangle ABC$, $\angle B$ is an obtuse angle and D is a point on CB produced such that AD is perpendicular to CB produced. Prove that $AC^2 = AB^2 + BC^2 + 2BC \times BD$.

Exercise 61. In $\triangle ABC$, D is a point on BC such that $AD \perp BC$. Prove that $BP^2 - PC^2 = BD^2 - DC^2$ for any point P on AD.

Exercise 62. In a $\triangle ABC$ with $\angle C = 90°$, P and Q are the mid points of CA and CB respectively. Prove that $4(AQ^2 + BP^2) = 5AB^2$.

Exercise 63. ABC is a triangle in which $AB = AC$ and D is any point on BC. Prove that $AB^2 - AD^2 = BD.CD$.

Exercise 64. An aeroplane leaves an airport and flies due north at a speed of 900 km per hour. At the same time, another aeroplane leaves the same airport and flies due west at a speed of 1200 km per hour. How far apart will be the two planes after $1\frac{1}{2}$ hours? (Answer. 2250 km.)

Exercise 65. ABC is a right triangle right angled at A. AD is the altitude through A; E is a point on AC such that $AE = CD$ and F is a point on AB such that $AF = BD$. Prove that $BE = CF$.

Exercise 66. ABC is a right triangle right angled at B in which $AB = 3$ cm and $BC = 4$ cm. D, E and F are points on BC such that $\angle BAD = \angle DAE = \angle EAF = \angle FAC$. Find BD, DE, EF and FC.

$$\left(\text{Answer. } BD = \frac{3}{2+\sqrt{5}},\ DE = \frac{3\sqrt{5}}{2(2+\sqrt{5})},\ EF = \frac{15\sqrt{5}}{2(10+3\sqrt{5})},\ FC = \frac{25}{10+3\sqrt{5}}.\right)$$

Exercise 67 (Carnot's theorem). Let D, E, F be points on the sides BC, CA, AB (or their extensions) respectively of a $\triangle ABC$. Prove that the perpendiculars to BC, CA, and AB at the points D, E, and F are concurrent if and only if $AF^2 + BD^2 + CE^2 = FB^2 + DC^2 + EA^2$.

Exercise 68 (Stewart's theorem). In $\triangle ABC$, let D be a point on BC such that $BD = m$ and $DC = n$. If $BC = a$, $CA = b$, $AB = c$ and $AD = d$, prove that $b^2m + c^2n = a(d^2 + mn)$.

the upper limit of a class is excluded from that class but an item equal to the lower limit of a class is included in that class.

Remark. The frequency of each class interval is assumed to be centred around its mid value. So, the mid value (or class mark) of each class is chosen to represent the observations falling in that class.

Do you know? One day, in 1939, a graduate student at the University of California, Berkeley arrived late for a class. He found two problems on the blackboard and assumed them to be homework problems. The problems seemed to be a little harder than usual, but a few days later he submitted the complete solutions for the problems. About six weeks later, one Sunday morning, he received a visit from the professor, who informed him that he had prepared one of his solutions for publication in a mathematical journal. The problems he had solved were in fact two famous unsolved problems in statistics. The professor had written the problems as examples of unsolved problems. The student was none other than George Dantzig, the man who later formulated the simplex method in linear programming. Linear programming is a mathematical technique for optimization of an outcome (such as maximizing profit and minimizing cost).

Definition 13.2. For a grouped frequency distribution having $x_1, x_2, \ldots, x_n$ as mid values of the classes with respective frequencies $f_1, f_2, \ldots, f_n$,

(a) the **weighted arithmetic mean** is the quantity $\overline{x}$ given by

$$\overline{x} = \frac{x_1 f_1 + x_2 f_2 + \cdots + x_n f_n}{f_1 + f_2 + \cdots + f_n} = \frac{1}{N}\sum_{i=1}^{n} x_i f_i, \quad \text{where } N = \sum_{i=1}^{n} f_i,$$

(b) the **weighted geometric mean**, denoted by G, is

$$G = \left(x_1^{f_1} x_2^{f_2} \cdots x_n^{f_n}\right)^{\frac{1}{f_1+f_2+\cdots+f_n}} = \left(\prod_{i=1}^{n} x_i^{f_i}\right)^{\frac{1}{N}}, \quad \text{where } N = \sum_{i=1}^{n} f_i,$$

(c) the **weighted harmonic mean**, denoted by H, is given by

$$\frac{f_1 + \cdots + f_n}{H} = \sum_{i=1}^{n} \frac{f_i}{x_i} \quad \textbf{or} \quad \frac{1}{H} = \frac{1}{N}\sum_{i=1}^{n} \frac{f_i}{x_i}, \quad \text{where } N = \sum_{i=1}^{n} f_i.$$

Remark. $A \geq G \geq H$, where A is the arithmetic mean, G is the geometric mean and H is the harmonic mean.

Limitations of GM and HM: The arithmetic mean is taken in most cases to represent the mean of the data mainly because of

(a) the inconvenience in calculating the numerical values of GM and HM,
(b) the lack of versatility of GM and HM.

Note that GM and HM become invalid when one of the variate values is zero. However, under special conditions GM may become the best mean and under other conditions HM may become the best mean of the data. In this chapter, we discuss only AM.

Notation. We use the notation (f_i, x_i), $i = 1, 2, \dots, n$, to denote a grouped frequency distribution having $x_1, x_2, \dots, x_n$ as mid values of the classes with respective frequencies $f_1, f_2, \dots, f_n$.

Methods for Finding Arithmetic Mean

Direct method: If the mid values (x_i's) and the frequencies (f_i's) are not very large, we can directly use the formula given in Definition 13.2 to find the arithmetic mean of a grouped frequency distribution.

Assumed mean or change of origin method: Let a be any convenient arbitrary number. Let $d_i = x_i - a$. Then $f_i d_i = f_i x_i - f_i a$. Summing both sides over i from 1 to n, we get

$$\sum_{i=1}^{n} f_i d_i = \sum_{i=1}^{n} f_i x_i - a \sum_{i=1}^{n} f_i = \sum_{i=1}^{n} f_i x_i - aN, \text{ where } N = \sum_{i=1}^{n} f_i$$

$$\implies \frac{1}{N} \sum_{i=1}^{n} f_i d_i = \frac{1}{N} \sum_{i=1}^{n} f_i x_i - a = \overline{x} - a$$

$$\implies \overline{x} = a + \frac{1}{N} \sum_{i=1}^{n} f_i d_i = a + \overline{d}, \quad \text{where } \overline{d} = \frac{1}{N} \sum_{i=1}^{n} f_i d_i.$$

It is more convenient if we take a to be a mid value which lies near the middle of the mid values of the distribution. The number a is called the assumed mean, each d_i is the deviation of a from x_i and $\overline{d}$ is the mean of the deviations.

Step deviation or change of origin and scale method: Let h be the uniform width of the class interval. We define a new variate u_i as follows.

$$u_i = \frac{x_i - a}{h}, \quad \text{where } a \text{ is the assumed mean.}$$

Proceeding in a similar way as in the assumed mean method, we get

$$\overline{x} = a + \frac{h}{N} \sum_{i=1}^{n} f_i u_i = a + h\overline{u}, \quad \text{where } \overline{u} = \frac{1}{N} \sum_{i=1}^{n} f_i u_i.$$

Solution. (Step deviation method) Taking $a = 55$, $h = 10$, $u_i = \dfrac{x_i - a}{h} = \dfrac{x_i - 55}{10}$, we obtained the following table.

Marks	Mid value x_i	$u_i = \dfrac{x_i - 55}{10}$	No. of students f_i	$f_i u_i$
$0 - 10$	5	-5	$(80 - 77) = 3$	-15
$10 - 20$	15	-4	$(77 - 72) = 5$	-20
$20 - 30$	25	-3	$(72 - 65) = 7$	-21
$30 - 40$	35	-2	$(65 - 55) = 10$	-20
$40 - 50$	45	-1	$(55 - 43) = 12$	-12
$50 - 60$	55	0	$(43 - 28) = 15$	0
$60 - 70$	65	1	$(28 - 16) = 12$	12
$70 - 80$	75	2	$(16 - 10) = 6$	12
$80 - 90$	85	3	$(10 - 8) = 2$	6
$90 - 100$	95	4	$(8 - 0) = 8$	32
			$N = 80$	$\sum f_i u_i = -26$

Now, $\overline{u} = \dfrac{1}{N} \sum f_i u_i = \dfrac{1}{80} \times (-26) = -0.325$.
Hence, the mean mark, $\overline{x} = a + h\overline{u} = 55 + 10 \times (-0.325) = 51.75$.

Definition 13.3 (Cumulative frequency). The cumulative frequency of a class is the frequency obtained by adding the frequencies of all the classes preceding the given class.

A cumulative frequency distribution is a table that shows the cumulative frequencies of the classes. There are two types of cumulative frequency distributions: (i) the cumulative frequency distribution of the less than type, and (ii) the cumulative frequency distribution of the more than type.

The cumulative frequency of the less than type for a particular value of the variate is obtained by adding the frequencies of all the values smaller than or equal to that value. On the other hand, the cumulative frequency of the more than type for a particular value is obtained by adding the frequencies of all the values greater than or equal to that value. These are illustrated in Table 1, page 443 and Table 2, page 443.

Definition 13.4 (Median). The median is the value of the variate such that half of the total number of variates have their values less than or equal to it and the other half have their values greater than or equal to it.

For a grouped frequency distribution, the *median class* is the class with cumulative frequency just greater than (and nearest to) half the size of

In Table 2 above, $0, 10, \ldots, 60$ are the lower limits of the respective classes. Plot the points $(0, 60)$, $(10, 56)$, etc., and join them by free hand to get the more than ogive (or cumulative frequency curve of the more than type).

The less than and the more than ogives of the given data are shown below.

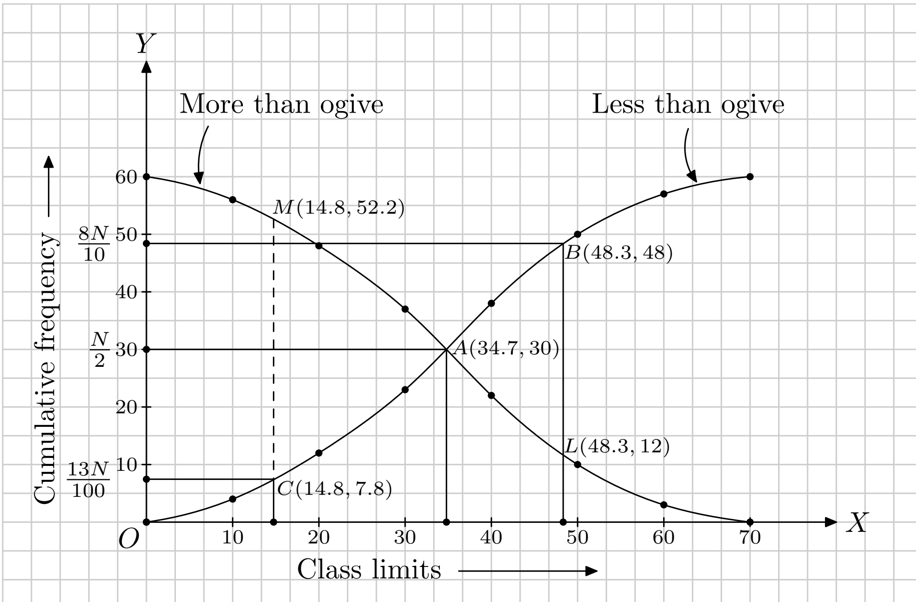

Figure 13.1: Ogives.

The less than ogive and the more than ogive of the given data intersect at a point A. Median is the x-coordinate of the point A. Mark points B and C on the less than ogive whose y-coordinates are $\frac{8N}{10}$ and $\frac{13N}{100}$ respectively. D_8 and P_{13} are the x-coordinates of the points B and C respectively. Also, D_8 and P_{13} are the x-coordinates of the points L and M (see figure above) on the more than ogive respectively. Note that the corresponding y-coordinates of the points L and M on the more than ogive are $\frac{(10-8)N}{10}$ and $\frac{(100-13)N}{100}$ respectively.

Exercise 13. The more than ogive curve and the less than ogive curve of a frequency distribution intersect each other at at the point $(30, 50)$.

(a) What is the median of the distribution? (Answer. 30.)

(b) What is the size of the population? (Answer. 100.)

(c) If the mean is 31, estimate the mode of the distribution.(Answer. 28.)

Exercise 14. If the mean and the median of a frequency distribution differs by 1.25, estimate the difference between the mean and the mode using Pearson's empirical formula. (Answer. 3.75.)

Exercise 15. Find the values of the unknown entries a, b, c, d, e, f, and hence find the mean, the median and the mode for the following frequency distribution.

Class	Frequency	Cumulative Frequency
0 – 8	15	a
8 – 16	b	28
16 – 24	15	c
24 – 32	d	61
32 – 40	e	70
40 – 48	10	f

Hint. $a = 15$, $b = 28 - a$, $c = 28 + 15$, $d = 61 - c$, $e = 70 - 61$, $f = 70 + 10$. Mean = 22.3. Median = 22.4. Mode = 26.

Exercise 16. Which measure of central tendency will be the most suitable in each of the following cases? Justify your answer in each case.

(1) To determine the productivity of a field using the data of the yield of the field for the past twenty years.

(2) To determine whether the literacy rate is the maximum in the age group 6 years to 14 years.

(3) To find the average of the marks obtained by the students in an examination.

(4) To find the average of the marks obtained by most of the students in an examination.

(5) To find the the mark above which only half the students scored in an examination.

(6) To find the typical productivity rate of workers.

(7) To find the average wage in a country.

(8) To find the most popular T.V. programme being watched.

(9) To determine the colour of the vehicle used by most of the people.

Hint. The measure of central tendency under study should possess the representative character of the data. Depending on the nature of the information that one is looking for, the appropriate measure of central tendency is to be fixed. Mean is suitable for (1), (3). Median is suitable for (5), (6), (7). There may be extreme values in (6) and (7). The mean is greatly affected by extreme values. So, rather than the mean, we take the median as a better measure of central tendency. Mode is suitable for (2), (4), (8), (9).

Hint. Out of n exhaustive, equally likely, mutually exclusive outcomes, if m are favourable to A, the remaining $n - m$ outcomes are not favourable to the event A.

Notes.

1. If there are n outcomes of a random experiment, then there are 2^n possible events.
2. When we say "independent events," we are referring to events with the same sample space. The same is the case with "equally likely events", "mutually exclusive events", "exhaustive events", etc.
3. In classical probability, we assume that all outcomes of a random experiment are equally likely.
4. The probability of an event may be an irrational number.
5. The sum of the probabilities of all the elementary events of an experiment is 1.
6. The sum of the probabilities of mutually exclusive events forming an exhaustive system is 1. In particular, if A, B and C are mutually exclusive and exhaustive events, then $P(A) + P(B) + P(C) = 1$.
7. If A and B are two events, then $A \cup B$ denotes the event of happening at least one of A and B, and AB (also denoted by $A \cap B$) denotes the event of the combined occurrence of A and B.
8. Two events A and B are said to be independent if and only if $P(AB) = P(A) \cdot P(B)$. Otherwise, A and B are called dependent events.
9. Mutually exclusive events have no common outcome.
10. Two independent events always have at least one common outcome if they are not impossible events.
11. (**Addition theorem**) If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B).$$

In general, if A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(AB).$$

Exercise 11. Why is tossing a coin considered to be a fair way of deciding which team should get the ball at the beginning of a football match?

Exercise 12. In two independent tosses of a fair die the sum of the outcomes was 9. What would be the probability that the first toss resulted in 6? (Answer. 1/4.)

Exercise 13. A bag contains 30 balls out of which some are red, some are blue and remaining are black. If the probability of drawing a red ball is 11/15 and that of a blue ball is 1/10, then how many black balls are there in the bag. (Answer. 5.)

Example 14. If a coin is tossed twice, the event of getting a head in the first toss and the event of getting a tail in the second toss are independent. Prove it.

Solution. The sample space is $\{HH, HT, TH, TT\}$. Let A be the event of getting a head in the first toss and B be the event of getting a tail in the second toss. Then $A = \{HH, HT\}$ and $B = \{HT, TT\}$. So, $AB = \{HT\}$. Now,

$$P(A) = \frac{2}{4} = \frac{1}{2}, \ P(B) = \frac{2}{4} = \frac{1}{2} \text{ and } P(AB) = \frac{1}{4}.$$

Clearly, we see that $P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(AB)$. This shows that A and B are independent. □

A note on playing cards: There are 52 cards in a pack (or deck) of cards which are divided into 4 suits of 13 cards each. The cards in each suit are Ace, King, Queen, Jack (or Knave), 10, 9, 8, 7, 6, 5, 4, 3 and 2. King, Queen and Jack are the face cards. Ace, King, Queen and Jack are the power cards. The colours and number of cards in each of the four suits in a pack are given below.

Suit name	Colour	Number of Cards			
		Ace	Face Cards	Numeral Cards	Total
Heart (♡)	Red	1	3	9	13
Diamond (♢)	Red	1	3	9	13
Club (♣)	Black	1	3	9	13
Spade (♠)	Black	1	3	9	13

Exercise 15. What is the probability of drawing a Queen from a pack of well-shuffled cards. (Answer. 1/13.)

Exercise 16. Give one example of each of the following:
(a) Two events which are mutually exclusive but not independent.
(b) Two events which are independent but not mutually exclusive.
(c) Two events which are neither independent nor mutually exclusive.
(d) Two events which are both independent and mutually exclusive.

Example 17. Khamba and Thoibi are friends. Find the probability that **(i)** they have different birthdays; **(ii)** they have the same birthday. Assume that they were not born on a leap year.

Solution. Khamba's birthday can be any day of the 365 days of the year. Also, Thoibi's birthday can be any day of the year. We assume that these 365 outcomes are equally likely.

(i) If Khamba's birthday is different from Thoibi's birthday, the number of favourable outcomes of Thoibi's birthday $= 365 - 1 = 364$.

Let E be the event that they have different birthdays. Then the probability that they have different birthdays, $P(E) = \frac{364}{365}$.

(ii) The event that they have the same birthday and the event that they have different birthdays are complementary. Hence, the probability that they have the same birthday $= P(\overline{E}) = 1 - P(E) = 1 - \frac{364}{365} = \frac{1}{365}$.

> ***Do you know?*** In a group of people, what is the probability that two of them have the same birthday? This is the birthday problem or birthday paradox. The probability is 100% when the number of people is more than 366 as there are only 366 possible birthdays. However, the probability reaches 99.9% with just 70 people and 50% with just 23 people. There is a cryptographic attack called the birthday attack that exploits the mathematics behind the birthday problem. Note that the birthday problem is different from finding the probability that, in a group of people, someone has the same birthday as you (or a particular person). In order to get more that 50% probability in this case, the number of people must be at least 253.

Example 18. If a leap year is selected at random, what is the probability that it will have 53 Sundays?

Solution. We know that a leap year has 366 days, i.e., 52 weeks and 2 days. These two days will be two consecutive days of a week. The sample space for the possible pairs of days is

$$\begin{aligned} S = \{&(\text{Sunday}, \text{Monday}), (\text{Monday}, \text{Tuesday}), (\text{Tuesday}, \text{Wednesday}), \\ &(\text{Wednesday}, \text{Thursday}), (\text{Thursday}, \text{Friday}), (\text{Friday}, \text{Saturday}), \\ &(\text{Saturday}, \text{Sunday})\}. \end{aligned}$$

Here, (Sunday, Monday) and (Saturday, Sunday) are the favourable cases. Therefore, the required probability $= \frac{\text{number of favourable cases}}{\text{total number of cases}} = \frac{2}{7}$.

Remark. In the Gregorian calendar, those years exactly divisible by 100, but not by 400, are not leap years. For example, the years 1700, 1800, and 1900 are not leap years, but the year 2000 is a leap year. A leap year

Chapter 15

Trading and Demat Account

Markets, like oceans, have turbulence. Some days the change in markets is very small, and some days it moves in a huge leap. Only fractals can explain this kind of random change.

— Benoit Mandelbrot

Investing involves setting aside money to generate future returns. Warren Buffett describes it as using money now to earn more later. The main goal is to grow wealth over time through various investment vehicles. The first step for a stock market investor is to open a Trading and Demat account.

Some Important Points

- The Securities and Exchange Board of India (SEBI) is the regulatory authority for the securities and commodity markets in India.

- To invest in shares and engage in stock trading, it is necessary to open a trading account with a SEBI-registered stockbroker and a demat account with a SEBI-registered Depository Participant.

- A trading account enables the buying and selling of shares from home or office, while a demat account stores shares in electronic form.

- A demat account stores shares in electronic form. Multiple demat accounts can be opened, and there are no charges for doing so.

- A demat account can have up to three account holders, and their names cannot be changed. A nil balance is allowed in a demat account.

High School Leaving Certificate Examination

Design of Question Paper for Mathematics X

Time: 3 hours. Full Marks: 80.

1. Weightage to objectives:

Objectives	K	U	A	S	Total
Percentage of marks	37	45	12	6	100
Marks	30	36	9	5	80

(K = Knowledge, U = Understanding, A = Application, S = Skill.)

2. Weightage to forms of questions:

Forms of questions	LA	SA_1	SA_2	SA_3	VSA	O	Total
Number of questions	5	3	6	5	8	5	32
Marks allotted	27	12	18	10	8	5	80
Estimated time (min)	70	33	36	20	13	8	180

(LA = Long Answer, SA = Short Answer, VSA = Very Short Answer, O = Objective.)

3. Weightage of content:

Unit	Name of the unit	Marks
I	Number System, Polynomials and Factorisation	14
II	Pair of Linear Equations in Two Variables, Quadratic Equations and AP	14
III	Triangles, Circles and Construction	15
IV	Trigonometric Ratios, Height and Distances and Coordiante Geometry	15
V	Mensuration	10
VI	Statistics and Probability	10
VII	Trading and Demat Account	2
	Total	80

4. Scheme of section: Nil.
5. Scheme of option: Internal option must be given in essay/long answer type questions testing the same objective.
6. Difficulty level: 20% difficult, 60% average, 20% easy.

Question Papers

2022
Mathematics
Full Marks - 80
Pass Marks - 20
Time : Three hours
Attempt all questions.
The figures in the right hand margin indicate the full marks for the questions.

For questions 1 to 5, write the letter corresponding to the correct answer.

1. If $x + 2$ is a factor of $x^3 + 3x^2 + 5x + k$, then the value of k is 1

(A) 6. **(B)** −6. **(C)** 30. **(D)** −30.

2. The sum and product of the roots of the quadratic equation $ax^2 + bx + c = 0$ are respectively 1

(A) $\frac{b}{a}$ and $\frac{c}{a}$. **(B)** $\frac{b}{a}$ and $-\frac{c}{a}$. **(C)** $-\frac{b}{a}$ and $\frac{c}{a}$. **(D)** $-\frac{b}{a}$ and $-\frac{c}{a}$.

3. PA and PB are tangent segments drawn from an external point P to a circle with centre O. If $\angle APB = 50°$, then $\angle AOB =$ 1

(A) 140°. **(B)** 130°. **(C)** 120°. **(D)** 110°.

4. The nth term of a sequence is given by $a_n = 4n - 5$. Which of the following statements about the sequence is true? 1

(A) It is not an AP.
(B) It is an AP with the first term 4 and common difference 5.
(C) It is an AP with the first term −5 and common difference 4.
(D) It is an AP with the first term −1 and common difference 4.

5. If A and B are independent events of a random experiment, then 1

(A) $P(A \cup B) = P(A) + P(B)$. **(B)** $P(AB) = P(A) + P(B)$.
(C) $P(A) = P(A) \cdot P(B)$. **(D)** $P(A) + P(B) = 1$.

6. When is an algebraic expression said to be a cyclic expression? 1

7. Name any real number x, if it exists, such that x^2 is not positive. 1

8. Write the statement of the converse of Pythagoras theorem. 1

9. How may terms are there in the finite AP: $7, 10, 13, \ldots, 49$? 1

10. If the coordinates of the end points of a diameter of a circle are $(-7, 3)$ and $(3, 5)$, write the coordinates of the centre of the circle. 1

11. The area of a circle is 1386 cm^2. Find its radius. 1

12. Write the expression for the area of a sector of a circle of radius r and sectorial angle θ measured in degrees. 1

13. When are events of a random experiment said to be equally likely? 1

14. Prove that $|a|^2 = a^2$ for any real number a. 2

15. Find the sum of all odd numbers lying between 20 and 60. 2

16. Taking a right triangle ABC, right angled at B, prove that $\cos^2 A + \sin^2 A = 1$. 2

17. The perimeter of a certain sector of a circle of radius 7 cm is 22 cm. Find the area of the sector. 2

18. In a T-20 cricket match, a batsman hits a boundary 3 times out of 14 balls he faced. Find the probability that he did not hit a boundary in a ball he faced. 2

19. Factorise $x(y^2 + z^2) + y(z^2 + x^2) + z(x^2 + y^2) + 3xyz$. 3

20. State and prove remainder theorem. 3

21. Solve graphically: 3

$$x + 2y = 8 \text{ and } 3x - y = 3.$$

22. Solve the quadratic equation $ax^2 + bx + c = 0$, by method of completing perfect square. 3

23. Prove that the tangents at any point of a circle is perpendicular to the radius through the point of contact.

24. Prove that $\dfrac{\cos\theta}{\operatorname{cosec}\theta + 1} + \dfrac{\cos\theta}{\operatorname{cosec}\theta - 1} = 2\tan\theta$. 3

25. State any four of the eleven field properties of the real numbers. 4

Or

If x, y, δ are real numbers and $\delta > 0$, prove that $|x-y| < \delta \iff y-\delta < x < y+\delta$. 4

26. Two stations A and B on a highway are 90 km apart. A car starts from A and another car starts from B at the same time. If they travel in the same direction they meet in 9 hours, but if they travel towards each other they meet in 1 hour after start. Find the speeds of the two cars, the car from A moving faster. 4

27. Prove that the coordinates of the point R which divides the line segment PQ joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ internally in the ratio $m : n$ are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right).$$ 4

28. State and prove SAS similarity theorem of triangles. 5

Or

Prove that the internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the other two sides. 5

29. Construct a pair of tangents to a circle from an external point. Write the steps of construction. $2+3=5$

30. The angle of elevation of the top of a tower from a point A on the ground is $30°$. On moving a distance of 20 m towards the foot of the tower to a point B, the angle of elevation increases to $60°$. Find the height of the tower and the distance of the tower from the point A. 5

Or

A tower subtends an angle α at a point on the same level as the foot of the tower and from a second point h meters above the first, the angle of depression of the foot of the tower is β. Show that the height of the tower in meters is $h \tan\alpha \cot\beta$. 5

31. A solid metallic sphere of radius 6 cm is melted and recast to form a cylinder of height 32 cm. Find the radius, curved surface area and total surface area of the cylinder. (Give answer in terms of π.) 6

32. Find the mean, the median and the mode of the following frequency distribution. 6

Class	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
Frequency	7	10	17	21	25	23	11	6

15. The sequence of all odd numbers between 20 and 60 is $21, 23, \ldots, 59$, which is an AP with $a = 21$, $d = 2$ and $l = 59$.
Let n be the number of term of the AP. Then we have

$$\begin{aligned} l &= a + (n-1)d \\ \implies 59 &= 21 + (n-1) \times 2 \\ \implies 2n - 2 &= 38 \\ \implies n &= 20. \end{aligned}$$

Now, we have

$$S_{20} = 21 + 23 + 25 + \cdots + 59 = \frac{n}{2}(a + l) = \frac{20}{2}(21 + 59) = 800.$$

Thus, the sum of all odd numbers between 20 and 60 is 800.

16. See the first Pythagorean relation on page 302.

17. See Q1 on page 368. Change the radius and the perimeter accordingly.

18. Let A be the event that the batsman did not hit a boundary in a ball he faced. It is given that he hit 3 boundaries out of 14 balls. Thus, he did not hit a boundary in $14 - 3 = 11$ balls. Hence, the probability that he did not hit a boundary in a ball he faced is

$$P(A) = \frac{\text{number of balls he did not hit a boundary}}{\text{total number of balls he faced}} = \frac{11}{14}.$$

19. See Q2(v) on page 73.

20. See the remainder theorem on page 47.

21. Se Q3(viii) on page 93.

22. See page 147 for solving the quadratic equation $ax^2 + bx + c = 0$, by method of completing perfect square.

23. See theorem 8.1 on page 261.

24. See Q11(vii) on page 312.

25. **1st problem:** See the field properties of the real numbers on page 23.

2nd problem: See property 6 on page 29.

26. See Q12 on page 139.

27. See the section formula on page 338.

8. Here, the first term $a = 12$ and the common difference $d = 12 - 9 = -3$. Therefore, the 11th term of the AP is $a_{11} = 12 + (11 - 1) \times (-3) = -18$.

9. **Pythagoras theorem:** In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

10. The angle of elevation of a point observed is the angle formed by the line of sight with the horizontal when the point being observed is above the horizontal through the eye.

11. Here, the circumference is 132 cm. If d is the diameter of the circle, then $\pi d = 132$. So, $d = \dfrac{132}{\pi} = 42$ cm.

12. Events of a random experiment are said to be mutually exclusive if the happening of one prevents the happening of all the others.

13. A demat account is an account that allows the investor to keep shares in a electronic form.

14. See Q5(iv) on page 33.

15. See Q6(iv) on page 161.

16. Putting $x = -a$ in $x^n + a^n$, we have

$$\begin{aligned} x^n + a^n &= (-a)^n + a^n \\ &= (-1)^n a^n + a^n \\ &= \{(-1)^n + 1\}\, a^n \\ &= 0 \text{ only when } n \text{ is odd.} \end{aligned}$$

Hence, $x^n + a^n$ is divisible by $x + a$ only when n is odd. □

17. See Q1 on page 245.

18. We have $\dfrac{\sin 18^\circ}{\cos 72^\circ} = \dfrac{\sin(90^\circ - 72^\circ)}{\cos 72^\circ} = \dfrac{\cos 72^\circ}{\cos 72^\circ} = 1$.

19. Since $x+2$ and $x+3$ are factors of $p(x) = x^3 + ax + b$, we have $p(-2) = 0$ and $p(-3) = 0$. So, we have

$$\begin{aligned} &(-2)^3 + a(-2) + b = 0 \text{ and } (-3)^3 + a(-3) + b = 0 \\ \implies &-8 - 2a + b = 0 \text{ and } -27 - 3a + b = 0. \end{aligned} \tag{1}$$

Subtracting the last two equations, we have

$$(-8 - 2a + b) - (-27 - 3a + b) = 0 \implies 19 + a = 0 \implies a = -19.$$

Putting the value of a in (1), we get $b = 2(-19) + 8 = -30$.

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Notations and Abbreviations

$\mathbb{N}$	the natural numbers
$\mathbb{Z}$	the integers
$\mathbb{Q}$	the rational numbers
$\mathbb{R}$	the real numbers
$\mathbb{C}$	the complex numbers
$\iff$	if and only if
$\implies$	implies
$\forall$	for all
$\exists$	there exists
$\triangle$	triangle
$\angle$	angle
$x \in A$	the element x belongs to the set A
$A \cup B$	the union of A and B
$A \cap B$	the intersection of A and B
$n!$	the product of first n natural numbers
$\lfloor x \rfloor$	the greatest integer less than or equal to x
$\lvert x \rvert$	the absolute value of x
(a, b)	the gcd of a and b (see Chapter 1)
$[a, b]$	the lcm of a and b
(x, y)	the coordinates of a point (see Chapter 11)
$\min\{a, b\}$	the minimum of a and b
$\max\{a, b\}$	the maximum of a and b
$\square$	QED (quod erat demonstrandum), that which was to be demonstrated
CMO	Canadian Mathematical Olympiad
IMO	International Mathematical Olympiad
INMO	Indian National Mathematical Olympiad
USAMO	United States of America Mathematical Olympiad

Index

STM

"A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas."

— G. H. Hardy, *A Mathematician's Apology*

About the Authors

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